Khovanov homology 101

Brent Everitt

Abstract. Lectures for the ICTP-AIMS workshop *Homological Methods in Algebra and Geometry*, Biriwa, Ghana August 2016.

Contents

Brent Everitt: Department of Mathematics, University of York, York YO10 5DD, United Kingdom. e-mail: brent.everitt@york.ac.uk.

1. Invariants

(work in the smooth category; equivalently in PL-category)

- link with n-components = sub-manifold $L \subset S^3$

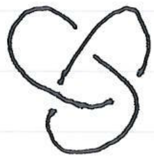
  homeomorphic to $\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{n}$ ($n=1$: knot)

  ambient isotopy (or equivalence) of links L, L'

  map $h: S^3 \times [0,1] \to S^3$ $\begin{cases} h_t \text{ homeo. } S^3 \to S^3 \\ h_0 = id \\ h_1(L) = L' \end{cases}$

- model combinatorially:

  links by diagrams = projections with crossing info.

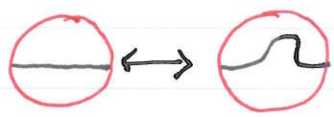


  (nice ones always exist)

  isotopies by movies = sequence of diagrams:

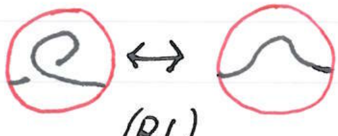
D	$\cdots$	D_i	D_{i+1}	$\cdots$	D'

s.t. each $D_i \to D_{i+1}$ is (0).



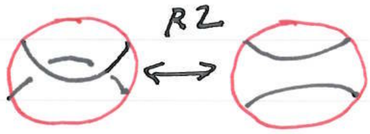
planar isotopy

(1)



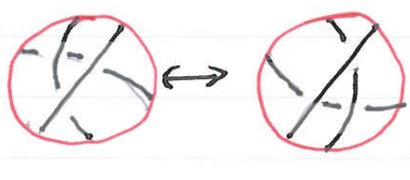
(R1)

(2).



R2

(3).



(R3)

$\underline{\text{Th}^{\underline{m}}}$ (Reidemeister): links L, L' equivalent $\Leftrightarrow$ any diagrams D, D' for them connected by a movie.

- <u>Invariants</u>: well defined labelling $L \mapsto i(L)$ of equivalence classes; i.e: L, L' equivalent $\Rightarrow i(L) = i(L')$.

  (<u>complete</u> invariant if an $\Leftrightarrow$)

- <u>Eg</u> (stupid): $i(L) = 0$ ($\in \mathbb{Z}$) for all L

  or $i(L) = L$ (complete...)

- <u>Eg</u>: homeomorphism type of $S^3 \setminus L$ (<u>complete</u> for knots)

- <u>Eg</u> (turns out to be stupid): "ordinary" homology

$$L \longmapsto S^3 \setminus L \longmapsto \tilde{H}_*(S^3 \setminus L)$$ (reduced) singular homology

Alexander duality $\Rightarrow \tilde{H}_{i-1}(S^3 \setminus L) \cong \tilde{H}^{3-i}(L) = \begin{cases} \mathbb{Z}^n, & i=2 \\ \mathbb{Z}^{n-1}, & i=3 \\ 0, & \text{else} \end{cases}$

(not very discerning...)

- <u>Eg</u>: homotopy: $L \longmapsto S^3 \setminus L \longmapsto \pi_*(S^3 \setminus L)$ homotopy groups

  $\widetilde{S^3 \setminus L}$ contractible $\Rightarrow \pi_i(S^3 \setminus L, x) = 0$ $(i > 1)$

  $\pi_1(S^3 \setminus L) :=$ <u>link group</u>

  <u>good news</u>: complete invariant for large classes of links (e.g: prime knots, hyperbolic knots)

bad news: ridiculously complicated Eg: $\pi_1(S^3 \setminus$  $, x)$ has a subgroup $\twoheadrightarrow$ every finite simple group!

— Eg: polynomial invariants (invariably computed via diagrams)

Kauffman bracket $\langle L \rangle \in \mathbb{Z}[q^{\pm 1}]$
$$\begin{cases} \langle \emptyset \rangle = 1 \\ \langle L \sqcup O \rangle = (q^{-1}+q)\langle L \rangle \quad (*) \\ \langle \text{crossing} \rangle = \langle \text{0-smoothing} \rangle - q \langle \text{1-smoothing} \rangle \end{cases}$$

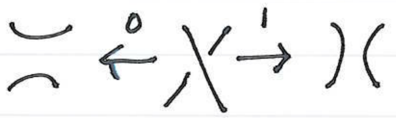
($\langle L \rangle$ uniquely determined by these axioms)

alternatively: $B(L)$ = poset of subsets of crossings of (diagram) of L ($\subseteq$ ordered)

write $\sigma \in B(L)$
$$\begin{cases} \sigma = \sigma_1 \cdots \sigma_n \text{ with } \sigma_i = 1 \Leftrightarrow i \in \sigma \\ \sigma \to \tau \text{ means } \sigma = \sigma_1 \cdots 0 \cdots \sigma_n \to \sigma_1 \cdots 1 \cdots \sigma_n = \tau \end{cases}$$

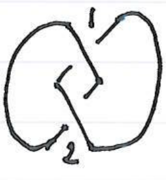
(so $\sigma \leq \tau \Leftrightarrow \sigma \to \sigma_1 \to \cdots \to \sigma_k \to \tau$)

crossing:



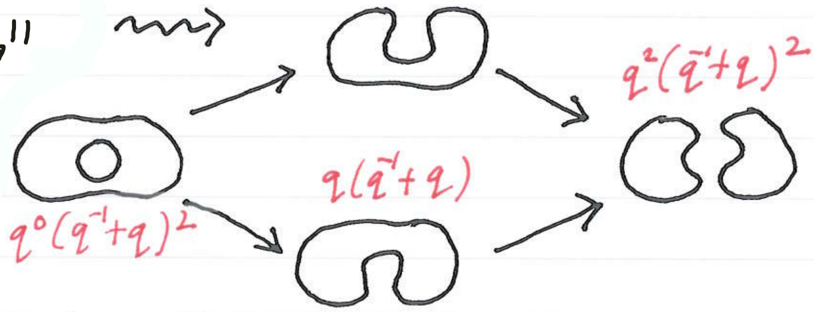
0,1-resolvings

$\sigma \rightsquigarrow$ smoothing of L = 0,1-resolve crossings according to σ



L

$\rightsquigarrow$ $B(L)$:
$$\begin{array}{ccccc} & & 10 & & \\ & \nearrow & & \searrow & \\ 00 & & & & 11 \\ & \searrow & & \nearrow & \\ & & 01 & & \end{array}$$
$\rightsquigarrow$



"cube" of smoothings

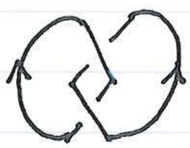
label: $\sigma \rightsquigarrow$ smoothing $\rightsquigarrow f_\sigma := q^{\Sigma \sigma_i}(q^{-1}+q)^{\# \text{circles}}$

then: $\langle L \rangle = \sum_\sigma (-1)^{\Sigma \sigma_i} f_\sigma$ (Eg: $\langle$  $\rangle$
$= q^{-2}+q^0+q^2+q^4$)

invariance: $\langle L \rangle$ computed via diagram, so $\langle - \rangle$ an invariant $\Leftrightarrow$ invariant under R-moves

but, $\langle$  $\rangle \neq \langle$  $\rangle$

- <u>oriented links</u>: $L^{\circlearrowleft} :=$ oriented version of L



 $\in L \rightsquigarrow$   $\in L^{\circlearrowleft}$

left $\# = l$ right $\# = r$

$L_1^{\circlearrowleft}$ and $L_2^{\circlearrowleft}$ equivalent $\overset{\text{def}}{\Leftrightarrow}$ orientation preserving ambient isotopy

(hence oriented R-moves; see Lecture 3)

- <u>Jones polynomial</u> $J(L^{\circlearrowleft}) := (-1)^l q^{r-2l} \langle L \rangle$ (<u>Ex</u>: J an invariant of oriented links)

$J($  $) = q^{-4}\langle L \rangle$

- <u>Natural</u>-ness:

$L_1 \xrightarrow{g} L_2$ "map of links" (i.e: $S^3 \setminus L_1 \xrightarrow{g} S^3 \setminus L_2$) $\rightsquigarrow$ π_* or $H_*(L_1) \xrightarrow{g_*} \pi_*$ or $H_*(L_2)$

induced morphism

whereas $J(L_1) \xrightarrow[\text{no sense!}]{\text{makes}} J(L_2)$

$$\begin{array}{ccc} \text{Jones} & & \pi_* \\ & \longleftrightarrow & \\ \text{not-natural} & & \text{natural} \\ \text{simple} & & \text{complex} \end{array}$$

<u>Khovanov</u> : naturalise Jones

(result: natural and simple)

2. Construction

THEME: "every object in mathematics is the Euler characteristic of some complex" - I. Frenkel (via Bar-Natan)

(all vector spaces over a fixed field F)

- **Graded spaces**: $\{A_i\}_{i\in\mathbb{Z}}$ spaces; $A = \bigoplus_{i\in\mathbb{Z}} A_i$ graded space

$x\in A_i$ has q-degree i in A.

Linear map $f: A\to B$ has degree m iff $f: A_i \mapsto B_{i+m}$ $(i\in\mathbb{Z})$

$A' \subseteq A$ graded subspace iff $A'_i = A'\cap A_i$

($\Rightarrow A/A'$ graded with $(A/A')_i = A_i/A'_i$)

A, B graded $\Rightarrow A\oplus B$, $A\otimes B$ graded with

$$(A\oplus B)_i = A_i\oplus B_i \qquad (A\otimes B)_i = \bigoplus_{k+l=i} A_k\otimes B_l$$

Eg: degree i part of $A\{k\}:=$

$$A\otimes \begin{array}{|c|l}\hline 0 & \\ \hline F & k \\ \hline 0 & \\ \hline \vdots & \end{array} \quad \text{is} \quad A_{i-k}\otimes F \cong A_{i-k} \qquad \begin{array}{|c|l}\hline A_{i-k+1} & i+1 \\ \hline A_{i-k} & i \\ \hline A_{i-k-1} & i-1 \\ \hline \end{array}$$

$$= \text{"q-shift"} \uparrow k$$

(graded) dimension $\dim A := \sum_{i\in\mathbb{Z}} \dim A_i\, q^i$

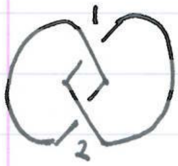
then $\dim A\oplus B = \dim A + \dim B$

$\dim A\otimes B = \dim A \dim B$ (finite dim's)

$\dim A\{k\} = q^k \dim A$

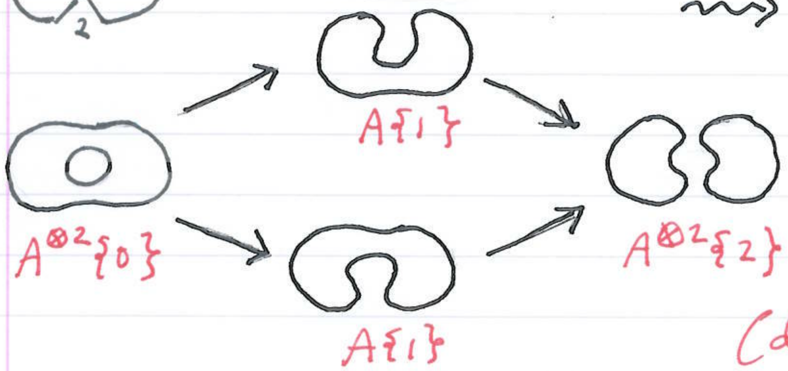
Eg (from now on) $A := \underset{-1}{F} \oplus \underset{1}{F} = F[1,u] = \begin{array}{|c|l}\hline 1 & 1 \\ \hline & \\ \hline u & -1 \\ \hline \end{array}$ $\quad \dim A = q^{-1}+q$

– L (unoriented) link, B(L) and smoothings from L1.

$2_1 =$ 

$\sigma \in B(L) \rightsquigarrow$ smoothing

$\rightsquigarrow A_\sigma := A^{\otimes \# \text{circles}}\{\Sigma \sigma_i\}$



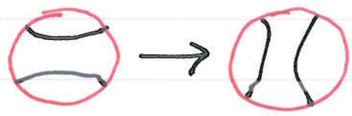
(precisely, associate copy of A to each circle and $\otimes$)

$(\dim A_\sigma = f_\sigma)$

– edge: $\sigma = \sigma_1 \cdots \underset{j}{0} \cdots \sigma_n \longrightarrow \sigma_1 \cdots \underset{j}{1} \cdots \sigma_n = \tau$

i.e: 

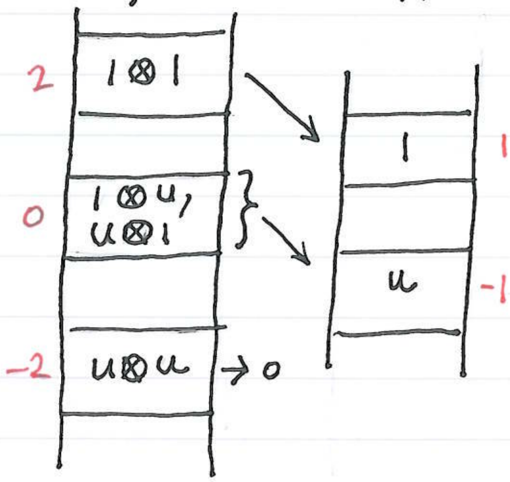
crossing j



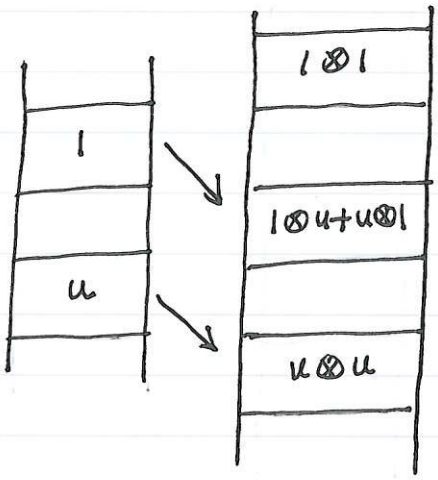
i.e: have either

define

$A^{\otimes 2} \xrightarrow{m} A$



$A \xrightarrow{\Delta} A^{\otimes 2}$



finally $A_\sigma \xrightarrow{d_\sigma^\tau} A_\tau$

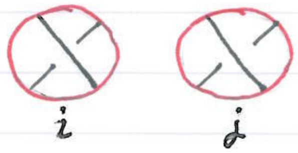
$= (m \text{ or } \Delta) \otimes (\text{id on other factors})$

$$\begin{tikzcd} & A\{1\} \arrow[dr, "\Delta"] & \\ A^{\otimes 2} \arrow[ur, "m"] \arrow[dr, "m"'] & & A^{\otimes 2}\{2\} \\ & A\{1\} \arrow[ur, "\Delta"'] & \end{tikzcd}$$

m, Δ degree $-1 \Rightarrow A^{\otimes k}\{\Sigma\sigma_i\} \xrightarrow[d_\sigma^\tau]{} A^{\otimes \ell}\{1+\Sigma\sigma_i\}$ degree 0

- square in $B(L)$:

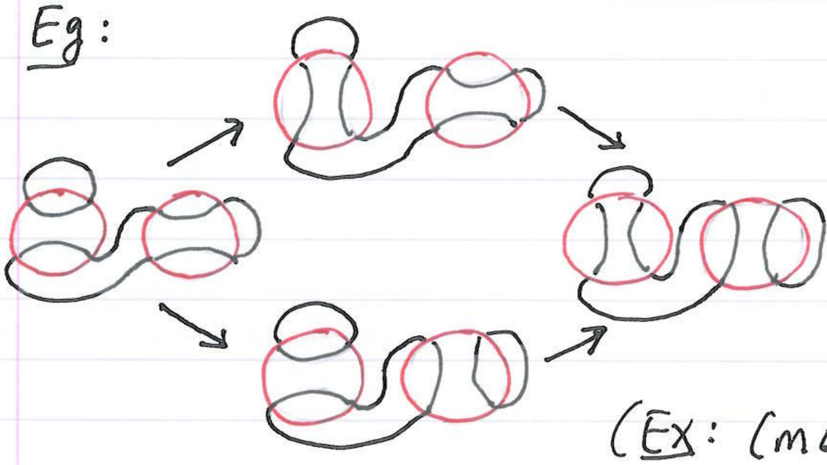
crossings i j



$$\begin{tikzcd} & 10 \arrow[dr] & \\ \underset{ij}{00} \arrow[ur] \arrow[dr] & & 11 \\ & 01 \arrow[ur] & \end{tikzcd}$$

(ignoring rest of string)

Eg:



$$\begin{tikzcd} & A \arrow[dr, "\Delta"] & \\ A\otimes A \arrow[ur, "m"] \arrow[dr, "1\otimes\Delta"'] & & A\otimes A \\ & A\otimes A\otimes A \arrow[ur, "m\otimes 1"'] & \end{tikzcd}$$

($\underline{\text{Ex}}$: $(m\otimes 1)(1\otimes\Delta) = \Delta m$)

$\Rightarrow$ squares commute.

- replace d_σ^τ by $(-1)^{\varepsilon_\sigma^\tau} d_\sigma^\tau$, $\varepsilon_\sigma^\tau = \sigma_1 + \cdots + \sigma_{j-1}$

$\sigma = \sigma_1 \cdots \underset{j}{0} \cdots \sigma_n$

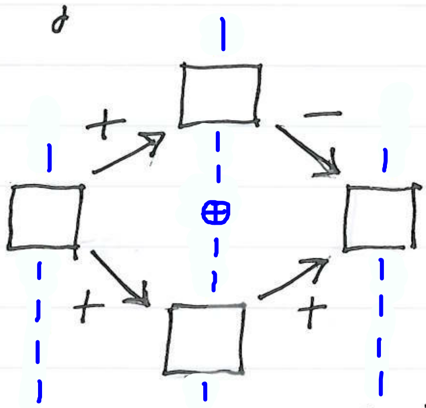
$\Rightarrow$ squares anticommute.

- $C^i(L) := \bigoplus_\sigma A_\sigma$ (sum over σ with $\Sigma\sigma_j = i$)

for $C^i(L) \xrightarrow{d} C^{i+1}(L)$

$s \in C^i$ and τ with $\Sigma\tau_j = i+1$

$ds \cdot \tau = \sum_\sigma (-1)^{\varepsilon_\sigma^\tau} d_\sigma^\tau (s.\sigma)$



$C^0(2_1) \to C^1(2_1) \to C^2(2_1)$

sum over the $\sigma = \tau_1 \cdots \hat{1} \cdots \tau_n$ $(\hat{1} = 0)$

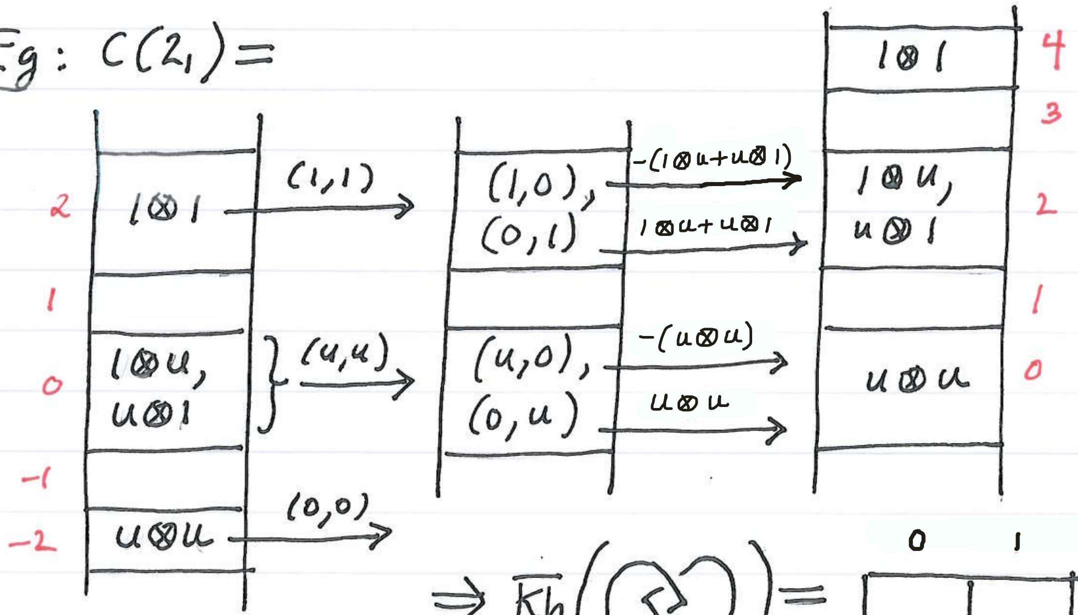
– (loosely) $d^2 = C^{i-1} \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} = \sum_{\text{squares}} \text{map} \curvearrowright + \text{map} \curvearrowright$ $= 0$

– (Unnormalised) <u>Khovanov</u> <u>homology</u> : $\overline{Kh}(L) := HC(L)$.

with $\sum_i (-1)^i \dim C^i(L) = \sum_\sigma (-1)^{\sum \sigma_i} f_\sigma = \langle L \rangle$

(Ex.) $\Rightarrow \chi \overline{Kh}(L) := \sum_i (-1)^i \dim \overline{Kh}^i(L) = \langle L \rangle$

– <u>Eg</u>: $C(2_1) =$



$\Rightarrow \overline{Kh}(\text{Hopf link}) =$

	0	1	2
4			F
3			
2			F
1			
0	F		
-1			
-2	F		

– <u>Ex</u>:

$\overline{Kh}(\bigcirc)$

	0
1	1
0	
-1	1

$\overline{Kh}(\text{Hopf link})$

	1
3	1
2	
1	1

$\overline{Kh}(\text{Hopf link})$

	0
0	1
-1	
-2	1

(just the dimensions from now on)

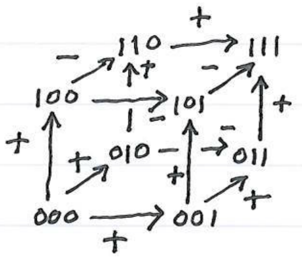
– <u>Khovanov homology</u>: $Kh(L^{\Omega}) := \overline{Kh}(L)\{-\ell\}[r-2\ell]$

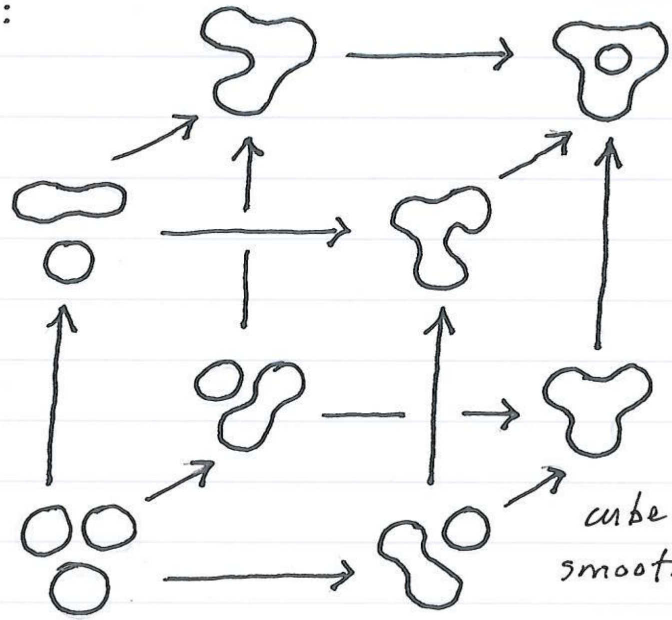
L^{Ω} has ℓ $\nwarrow\!\!\nearrow$'s and r $\nwarrow\!\!\nearrow$'s

$\Rightarrow \chi Kh(L^{\Omega}) = (-1)^{\ell} q^{r-2\ell} \langle L \rangle = J(L^{\Omega})$.

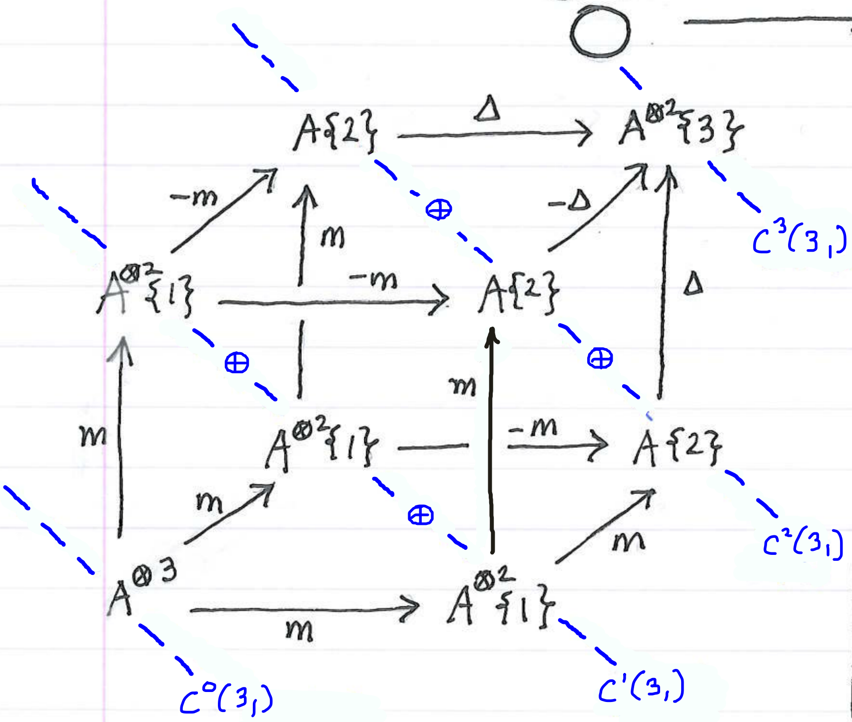
- Eg: Kh() :

3_1



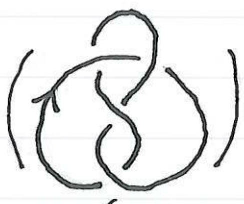


cube of smoothings



$\Rightarrow Kh(3_1) =$

	−3	−2	−1	0
−1				1
−2				
−3				1
−4				
−5		1		
−6				
−7				
−8				
−9	1			

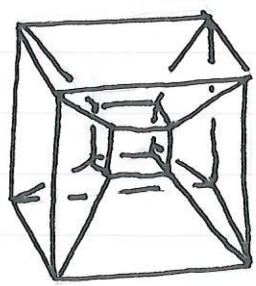
- Ex: Kh()

4_1

(and observe that $Kh^{i,j}(4_1)$

$\cong Kh^{-i,-j}(4_1)$

c.f. Lecture 5)

hint: 

3. Invariance

- **Theorem**: L_1, L_2 oriented links with L_1 equiv. to L_2

$$\Rightarrow Kh(L_1) \cong Kh(L_2).$$

- **need**: $Kh(\;) \cong Kh(\;) \cong Kh(\;)$

$Kh(\;) \cong Kh(\;) \cong Kh(\;)$ and $Kh(\;) \cong Kh(\;)$

$$Kh(\;) \cong Kh(\;)$$

- **Eg**: $Kh(\;) \cong Kh(\;) = \overline{Kh}(\;)[-\ell]\{r-2\ell\}$

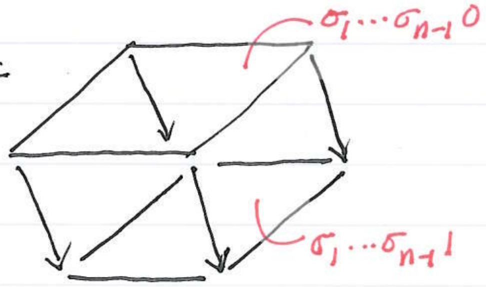
$\quad \|$

$Kh(\;)[-\ell]\{r-2\ell+1\}$ $\qquad$ $\overset{r}{\ell} \Rightarrow \overset{r+1}{\ell}$

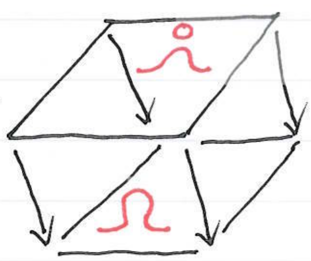
thus:

$$Kh(\;) \cong Kh(\;) \Leftrightarrow \overline{Kh}(\;) \cong \overline{Kh}(\;)\{-1\}$$

- $B(\underset{n}{\;}) =$



$\Rightarrow$ cube smoothings $=$



$$\Rightarrow C(\;) = \cdots \longrightarrow C^i(\;) \xrightarrow{d} C^{i+1}(\;) \longrightarrow \cdots$$

$$= \begin{cases} \cdots \longrightarrow C^i(\;) \xrightarrow{d_0} C^{i+1}(\;) \longrightarrow \cdots \\ \qquad\quad \oplus \qquad\quad \searrow m \qquad\quad \oplus \\ \cdots \longrightarrow C^{i-1}(\Omega)\{1\} \xrightarrow{d_1} C^i(\Omega)\{1\} \longrightarrow \cdots \end{cases}$$

where $s \in C^i(\;)$ is $s = (s.0, s.1)$ and

$$s.0 \in C^i(\;) = \bigoplus_{\sigma_1 \cdots \sigma_{n-1} 0} A \otimes A_{\sigma_1 \cdots \sigma_{n-1}} \quad (\Sigma \sigma_j = i)$$

$$s.1 \in C^{i-1}(\Omega)\{1\} = \bigoplus_{\sigma_1\cdots\sigma_{n-1}1} A_{\sigma_1\cdots\sigma_{n-1}}\{1\} \quad (\Sigma\sigma_j = i-1)$$

"m" means the sum of the $A \otimes A_{\sigma_1\cdots\sigma_{n-1}} \xrightarrow{m} A_{\sigma_1\cdots\sigma_{n-1}}\{1\}$

and $ds = (ds.0, ds.1) = (d_0(s.0), m(s.0) + d_1(s.1))$

– $C^i(\overset{01}{\Omega}) \overset{def}{=} \bigoplus_{\sigma 0} \langle 1 \otimes a_\sigma \rangle$ with $1 \otimes a_\sigma \in A \otimes A_\sigma$

sum over the $\sigma = \sigma_1 \cdots \sigma_{n-1}$ with $\Sigma\sigma_j = i$

$s \in C^i(\overset{01}{\Omega})$ write $s = (1 \otimes s.\sigma)_\sigma$ a σ-tuple

or $1 \otimes (s.\sigma)$ for short.

$$\Rightarrow D = \left\{ \begin{tikzcd} \cdots \arrow[r] \arrow[dr] & C^i(\overset{01}{\Omega}) \arrow[r, "d_0"] \arrow[dr, "m"] \arrow[d, phantom, "\oplus"] & C^{i+1}(\overset{0}{\Omega}) \arrow[r] \arrow[dr] & \cdots \\ \cdots \arrow[r] & C^{i-1}(\Omega)\{1\} \arrow[r, "d_1"'] & C^i(\Omega)\{1\} \arrow[r] & \cdots \end{tikzcd} \right.$$

a sub-complex of $C(\Omega)$

$$= \left\{ \begin{tikzcd} \cdots \arrow[r] & C^i(\overset{01}{\Omega}) \arrow[r, "1\otimes d_1"] \arrow[dr, "\mp\varepsilon"'] & C^{i+1}(\overset{01}{\Omega}) \arrow[r] \arrow[dr, "\pm\varepsilon"] \arrow[d, phantom, "\oplus"] & \cdots \\ & \cdots \arrow[r] & C^i(\Omega)\{1\} \arrow[r, "d_1"'] & C^{i+1}(\Omega)\{1\} \arrow[r] & \cdots \end{tikzcd} \right.$$

with $\varepsilon : 1 \otimes (s.\sigma) \mapsto (s.\sigma)$

$d_0 = 1 \otimes d_1 : 1 \otimes (s.\sigma) \mapsto 1 \otimes (d_1 s.\sigma)$

– $s \in D^{i+1}$ <u>cocycle</u> with $s = (s.0, s.1)$

$$\Rightarrow 0 = ds.1 = \pm\varepsilon(s.0) + d_1(s.1) \Rightarrow \overbrace{d_1(s.1) = \mp\varepsilon(s.0)}^{(*)}$$

Let $t \in D^i$ with $t \cdot 0 = 1 \otimes (\mp s \cdot 1)$
$t \cdot 1 = 0$

$\Rightarrow dt \cdot 1 = \mp \varepsilon(t \cdot 0) + d_1(t \cdot 1) = \mp \varepsilon(t \cdot 0) = \mp \varepsilon(1 \otimes (\mp s \cdot 1))$

$= s \cdot 1$

$dt \cdot 0 = d_0(t \cdot 0) = (1 \otimes d_1)(1 \otimes (\mp s \cdot 1))$

$= \mp(1 \otimes d_1(s \cdot 1)) \overset{(*)}{=} \mp(1 \otimes \mp \varepsilon(s \cdot 0)) = s \cdot 0$

$\Rightarrow dt = s \Rightarrow HD^* = 0$ (top/bottom degrees slightly different)

– $C/D \cong \left\{ \begin{array}{l} \cdots \to C^i(\frown)\{-1\} \to C^{i+1}(\frown)\{-1\} \to \cdots \\ \qquad\qquad \oplus \qquad\qquad\qquad \oplus \\ \cdots \to \quad 0 \quad\longrightarrow\quad 0 \to \cdots \end{array} \right.$

as $C^i(\overset{0}{\frown}) / C^i(\overset{01}{\frown}) = \bigoplus_{\sigma 0} A \otimes A_\sigma / \langle 1 \otimes a_\sigma \rangle$

and $\begin{array}{l} 1 \otimes x \\ u \otimes x \end{array} \mapsto \begin{array}{l} 0 \\ x \end{array}$ induces $A \otimes A_\sigma \to A_\sigma\{-1\}$

hence isom. $A \otimes A_\sigma / \langle 1 \otimes a_\sigma \rangle \xrightarrow{\cong} A_\sigma\{-1\}$

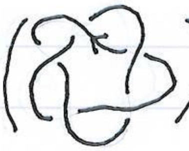
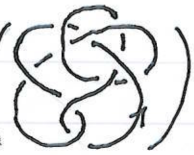
i.e: $C/D \cong C(\frown)\{-1\}$

– $\overline{Kh}(\wp) = HC(\wp) \cong H(C/D) \cong HC(\frown)\{-1\} = \overline{Kh}(\frown)\{-1\}$
as required.

– Ex: show others (or read [BN02])

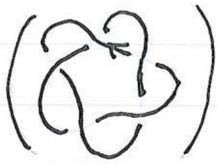
– L_1, L_2 oriented links with $J(L_1) \neq J(L_2) \Rightarrow Kh(L_1) \neq Kh(L_2)$

i.e: Kh at least as strong an invariant as Jones good news:

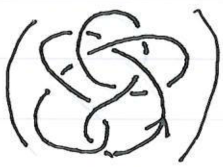
- Eg: $J\Big($  $\Big) = J\Big($  $\Big) = -q^{-15} + q^{-7} + q^{-5} + q^{-3}$

5_1 10_{132}

	-5	-4	-3	-2	-1	0
-3						1
-5						1
-7				1		
-9						
-11		1	1			
-13						
-15	1					

$= Kh\Big($  $\Big)$

	-7	-6	-5	-4	-3	-2	-1	0
-1							1	1
-3								1
-5					1	2		
-7				1				
-9				1	1			
-11		1	1					
-13								
-15	1							

$= Kh\Big($  $\Big)$

... even better news:

- there is a slight variation, <u>reduced</u> Khovanov homology

$\widetilde{Kh}(L)$, an invariant with $\widetilde{Kh}(\bigcirc) = \begin{array}{c} 0 \\ \boxed{1} \end{array} \; 0$

and

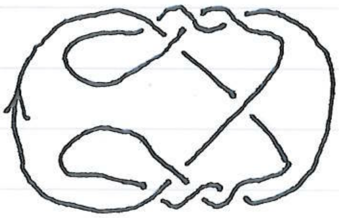
$\widetilde{Kh}(\underset{\substack{\uparrow \\ n \text{ component link}}}{L}) \cong \widetilde{Kh}(\underbrace{\bigcirc \cdots \bigcirc}_{n}) \Rightarrow L = \underbrace{\bigcirc \cdots \bigcirc}_{n}$

($n=1$ [KM11]; $n>1$ [HN13]).

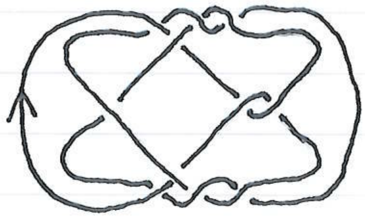
- (c.f.) Conjecture: $J(\underset{\text{knot}}{L}) = J(\bigcirc) \Rightarrow L = \bigcirc$

but $J(L) = J(\underbrace{\bigcirc \cdots \bigcirc}_{n>1}) \not\Rightarrow L = \bigcirc \cdots \bigcirc$

- not such good news:



8_8



10_{129}

$$Kh(8_8) \cong Kh(10_{129}) \text{ but } 8_8 \not\simeq 10_{129}.$$

(see [Wat07]).

4. Calculations

- Skein relation $\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$ for Kauffman bracket $\Rightarrow$ inductive (on # of crossings) calculations.

Khovanov homology has a long exact sequence.

- Analogously to $C(\mathcal{L})$ in §3:

$$C(\times) = \left\{ \begin{tikzcd} \cdots \arrow[r] \arrow[ddr] & C^{k}(\asymp) \arrow[r] \arrow[ddr] & \cdots \\ & \oplus & \\ \cdots \arrow[r] & C^{k-1}()()\{1\} \arrow[r] & \cdots \end{tikzcd} \right.$$

(q-grading $\perp$ to page)

$\Rightarrow C^{*-1}()()\{1\}$ sub-complex of $C^*(\times)$ with quotient $\cong C^*(\asymp)$, i.e: we have short exact sequence (#0):

$$0 \to C^{*-1}()()\{1\} \to C^*(\times) \to C^*(\asymp) \to 0$$

All maps are (q-degree) 0, so "fixing a height off the page":

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ 0 \arrow[r] & C^{k,p-1}()() \arrow[r] \arrow[u] & C^{k+1,p}(\times) \arrow[r] \arrow[u] & C^{k+1,p}(\asymp) \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & C^{k-1,p-1}()() \arrow[r] \arrow[u] & C^{k,p}(\times) \arrow[r] \arrow[u] & C^{k,p}(\asymp) \arrow[r] \arrow[u] & 0 \\ & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \end{tikzcd}$$

i.e: short exact sequence (#1):

$$0 \to C^{*-1,p-1}()() \to C^{*,p}(\times) \to C^{*,p}(\asymp) \to 0$$

$\Rightarrow$ long exact sequence (LES $\times$):

$$\cdots \to \overline{Kh}^{k-1,p}(\asymp) \to \overline{Kh}^{k-1,p-1}()() \to \overline{Kh}^{k,p}(\times) \to \overline{Kh}^{k,p}(\asymp) \to \overline{Kh}^{k,p-1}()() \to \cdots$$

- L^{o} = L oriented with $l = \#\, \nwarrow\!\!\nearrow$'s, $r = \#\, \nwarrow\!\!\nearrow$'s

$$\overline{Kh}(L) = Kh(L^{o})[l]\{2l-r\} \Rightarrow \overline{Kh}^{k,p}(L) = Kh^{k-l,\, p+r-2l}(L^{o})$$

- <u>case 1</u>: $\times$ in L becomes $\nwarrow\!\!\nearrow$ in L^{o}

$\Rightarrow$ $)($ becomes $\uparrow\uparrow$ with $l(\uparrow\uparrow) = l(\nwarrow\!\!\nearrow) - 1$

$r(\uparrow\uparrow) = r(\nwarrow\!\!\nearrow)$

$\asymp$ doesn't inherit orientation; let $\asymp^{o} = \asymp$ oriented some way.

with $l(\asymp^{o}) = l(\nwarrow\!\!\nearrow) + c$ $(c \in \mathbb{Z})$.

$\Rightarrow r(\asymp^{o}) = \#\text{crossings}\asymp - l(\asymp^{o})$

$$= (r(\nwarrow\!\!\nearrow) + l(\nwarrow\!\!\nearrow) - 1) - (l(\nwarrow\!\!\nearrow) + c) = r(\nwarrow\!\!\nearrow) - c - 1$$

plugging it all into LES $\times$ (and relabelling indices):

$$\boxed{\begin{gathered}\cdots \to Kh^{i,j+1}(\uparrow\uparrow) \to Kh^{i,j}(\nwarrow\!\!\nearrow) \to Kh^{i-c,\, j-3c-1}(\asymp^{o}) \to \cdots \\ c = l(\asymp^{o}) - l(\nwarrow\!\!\nearrow)\end{gathered}}$$

LES $\nwarrow\!\!\nearrow$

- <u>case 2</u>: $\times$ in L becomes $\swarrow\!\!\nwarrow$ in L^{o} gives:

$$\boxed{\begin{gathered}\cdots \to Kh^{i-d,\, j-3d-2}()(^{o}) \to Kh^{i,j}(\swarrow\!\!\nwarrow) \to Kh^{i,j-1}(\leftleftarrows) \to \cdots \\ d = l()(^{o}) - l(\swarrow\!\!\nwarrow)\end{gathered}}$$

LES $\swarrow\!\!\nwarrow$

- **Eg**: L has an odd (resp. even) # components

$$\Rightarrow Kh^{*,\text{even}} \ (\text{resp. } Kh^{*,\text{odd}}) = 0.$$

($\Rightarrow Kh^{*,\text{even}}(L) = 0$ for L a knot).

first: - **Ex**: with c, d as in LES's; in and if two strands (i). same component $\Rightarrow \begin{cases} c \text{ odd} \\ d \text{ even} \end{cases}$

(ii). different $\Rightarrow \begin{cases} c \text{ even} \\ d \text{ odd} \end{cases}$

- **Ex**: (i). $C(L_1 \sqcup L_2) = C(L_1) \otimes C(L_2)$

(ii). $\xRightarrow{\text{künneth}} \overline{Kh}(L_1 \sqcup L_2) \cong \overline{Kh}(L_1) \otimes \overline{Kh}(L_2)$

(iii). $\Rightarrow Kh$ similarly

$$\Rightarrow Kh^{i,j}(L_1 \sqcup L_2) \cong \bigoplus_{\substack{p+q=i \\ s+t=j}} Kh^{p,s}(L_1) \otimes Kh^{q,t}(L_2)$$

- back to **Eg**: induction on $m = $ # crossings ($n =$ # components)

(i). $m = 0 \Rightarrow L = \underbrace{\bigcirc \cdots \bigcirc}_{n}$ with $Kh(\bigcirc) = \underset{-1}{F} \oplus \underset{1}{F}$ $(= A)$

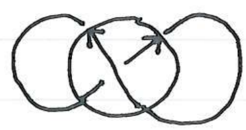
($\Rightarrow$ result for $n=1$) and

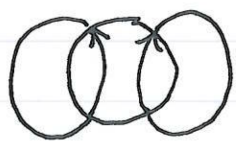
$$Kh(\underbrace{\bigcirc \cdots \bigcirc}_{n}) \cong Kh(\underbrace{\bigcirc \cdots \bigcirc}_{n-1}) \otimes Kh(\bigcirc)$$

$$\Rightarrow Kh^{i,j}(\underbrace{\bigcirc \cdots \bigcirc}_{n}) \cong Kh^{i,j-1}(\underbrace{\bigcirc \cdots \bigcirc}_{n-1}) \oplus Kh^{i,j+1}(\underbrace{\bigcirc \cdots \bigcirc}_{n-1})$$

$\Rightarrow$ result.

(ii). $m > 0$: choose a crossing $\Rightarrow$ four cases depending on whether or and whether strands $\in$ same/diff. components.

Suppose and same $\Rightarrow L =$  and c odd

$\Rightarrow$  $n+1$ components $m-1$ crossings and  n components $m-1$ crossings

with

$$\cdots \to Kh^{i,j+1}(\;) \to Kh^{i,j}(\;) \to Kh^{i-c,\,j-3c-1}(\;) \to \cdots$$

m odd/even and j even/odd $\Rightarrow$ $j+1$ odd/even

$j-3c-1$ even/odd

$\xRightarrow{\text{induction}}$ $Kh^{i,j+1}(\;) = 0 = Kh^{i-c,\,j-3c-1}(\;)$

$\xRightarrow{\text{exactness}}$ $Kh^{i,j}(\;) = 0$.

5. Natural-ness

- $\underline{Vect_F}$ = category of vector spaces over F:

  - objects: (finite dim) vector spaces over F

  - morphisms: linear maps

- $\underline{Links}$: category with

  - objects: oriented links in S^3 (including empty link)

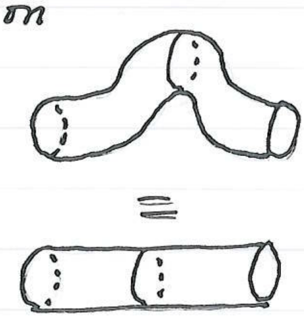
  - morphisms: $\Sigma : L_1 \to L_2$ $\underline{cobordism}$

    = compact oriented surface in $S^3 \times [0,1]$

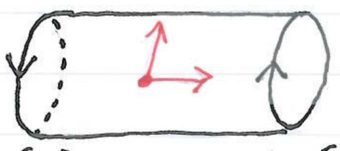
    with $\partial\Sigma = (-L_1 \times \{0\}) \cup (L_1 \times \{1\})$

    ↑ opposite orientation

    $\underline{upto}$ isotopy keeping $\partial\Sigma$ fixed:

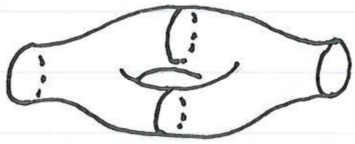


$\underline{Eg}$:



$-L\times\{0\}$ $\quad$ $L\times\{1\}$

isotopy unknot L $\to$ unkot L



another $\Sigma : L \to L$

isotopy (or equivalence) $L \to L'$ gives cobordism.

– modelling Links combinatorally: links : link diagrams
cobordisms : movies !

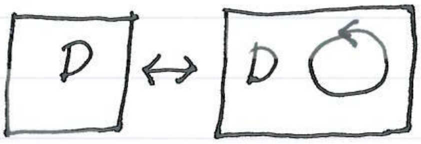
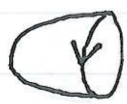
(extend Lecture 1 movies)

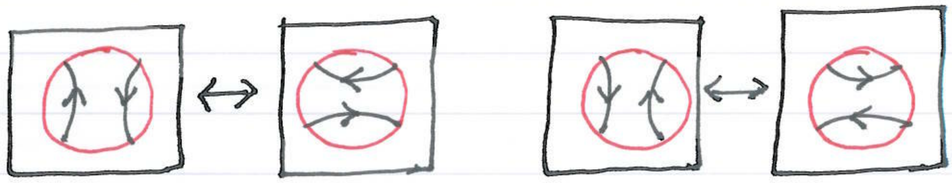
$M =$

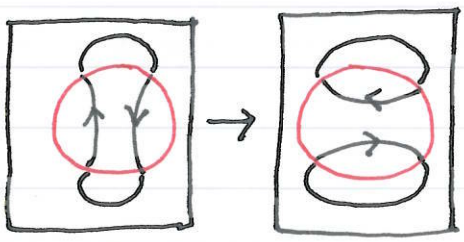
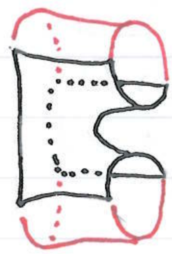
D	$\cdots$	D_i	D_{i+1}	$\cdots$	D'

s.t. each $D_i \to D_{i+1}$ is (1) an R-move

$\uparrow$ isotopy

(2).  (models : )

(3). 

Eg:  models : 

just as: $L \to L'$
isotopy of links

$\Updownarrow$

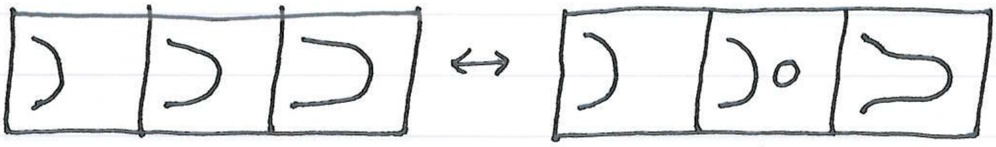
$D \to D'$
R-moves

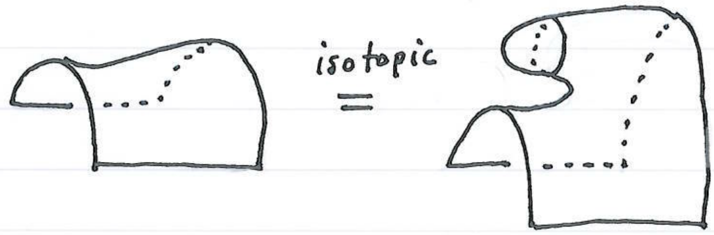
so: $\Sigma \to \Sigma'$
isotopy of cobordisms

$\Updownarrow$

$M \to M'$
movie moves

Eg (movie move):

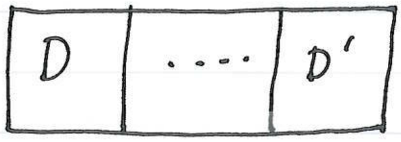


(models: )

- Theorem: Khovanov homology extends to a (covariant) functor $Kh: \underline{Links} \to \underline{Vect}_{\mathbb{F}_2}$ s.t. $\Sigma: L \to L'$ cobordism then $Kh(\Sigma): Kh(L) \to Kh(L')$ map of bidegree $(0, \chi(\Sigma))$ (i.e: $Kh(\Sigma): Kh^{i,j}(L) \to Kh^{i,j+\chi(\Sigma)}(L')$)

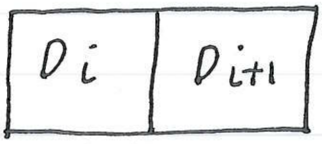
(take $Kh(\emptyset) = \boxed{\overset{\circ}{\mathbb{1}}}$)

- defining $Kh(\Sigma)$:



movie for Σ...

... for each frame:



define $Kh(D_i) \to Kh(D_{i+1})$ using (1)–(3) above...

... and compose!

(e.g: if $D_i \xrightarrow{R\text{-move}} D_{i+1}$ have $Kh(L_i) \xrightarrow{\cong} Kh(L_{i+1})$)

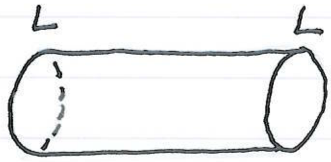
remark: some movie moves change sign of $Kh(\Sigma)$, hence work over $\mathbb{F}_2$.

- Eg (using naturalness): if $L^!$ = mirror image of (oriented) L then $Kh^{i,j}(L^!) \cong Kh^{-i,-j}(L)$


L


$L^!$

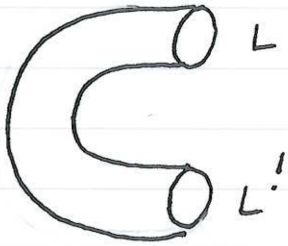
in Links $L \times [0,1] =$



identity cobordism $L \to L$

$\xrightarrow{\text{bend}}$

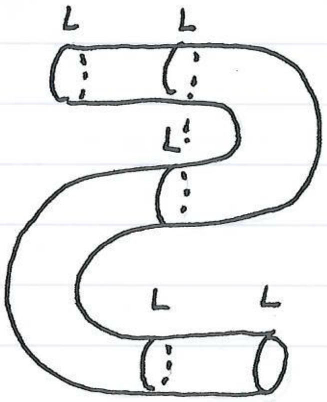
and



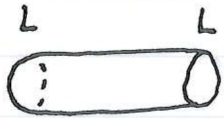
$\Sigma : L^! \sqcup L \to \phi$ $\Sigma' : \phi \to L \sqcup L^!$

cobordisms

with



$\overset{\text{straighten}}{=}$



$L = \phi \sqcup L \to L \sqcup \phi = L$

$$\xrightarrow[\text{everything in sight}]{\text{apply Kh functor to}}$$

$Kh(\Sigma) : Kh(L^!) \otimes Kh(L) \to \mathbb{F}_2$ $Kh(\Sigma') : \mathbb{F}_2 \to Kh(L) \otimes Kh(L^!)$

with

$$\begin{tikzcd}
Kh(L) \arrow[rr, "1"] \arrow[d, "\cong"'] & & Kh(L) \\
\mathbb{F}_2 \otimes Kh(L) \arrow[dr, "Kh(\Sigma')\otimes 1"'] & & Kh(L) \otimes \mathbb{F}_2 \arrow[u, "\cong"'] \\
& Kh(L) \otimes Kh(L^!) \otimes Kh(L) \arrow[ur, "1 \otimes Kh(\Sigma)"'] &
\end{tikzcd}$$

commuting

$\Rightarrow$ pairing $Kh(\Sigma): Kh(L^!) \otimes Kh(L) \to \mathbb{F}_2$ non-degenerate

put degrees back in: $Kh(\emptyset) = \overset{0}{\boxed{\mathbb{F}_2}}$

$\Rightarrow$ map:

$$\left(Kh(L^!) \otimes Kh(L)\right)_{0,0} = \bigoplus_{i,j} Kh^{i,j}(L^!) \otimes Kh^{-i,-j}(L)$$

$$\to \mathbb{F}_2$$

$\Rightarrow$ (fixed i,j) map: $Kh^{i,j}(L^!) \otimes Kh^{-i,-j}(L) \to \mathbb{F}_2$

non-degenerate by above $\Rightarrow$ $\dim Kh^{i,j}(L^!)$

$= \dim Kh^{-i,-j}(L)$.

Appendix: Complexes 101

The following is a non-encyclopedic summary of the basic notations and operations on cochain complexes that we will need. The *algebraic* difference between chain complexes and cochain complexes are mainly in indexing, and so we will just stick to cochain terminology. Standard references for these (and much else) are [Rot09, Wei94].

Complexes are hybrid algebro-topological objects. On the algebra side there are subcomplexes, homomorphisms (that have kernels and images), quotients, isomorphism theorems, etc. In other words, they are very much like vector spaces or Abelian groups, or more generally, R-modules. On the topological side you can say when two maps between complexes are homotopic, when two complexes are homotopy equivalent, etc. So they are also like topological spaces.

There is one operation special to complexes, and that is taking cohomology.

Complexes: A *(cochain) complex* $C = C^*$ consists of a family $\{C^n\}_{n\in\mathbb{Z}}$ of vector spaces/Abelian groups/R-modules – choose your preferred level of generality – and a family $\{d^n : C^n \to C^{n+1}\}_{n\in\mathbb{Z}}$ of homomorphisms, normally written out as

$$\cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \to \cdots$$

and such that for each n, we have the image of the map d^{n-1} is contained in the kernel of the map d^n. Equivalently, $d^n d^{n-1} = 0$.

In these notes the C^n will always be finite dimensional vector spaces over some field F and the $C^n \neq 0$ for finitely many n. We then write

$$C^k \to \cdots \to C^n \to \cdots C^\ell$$

The C^n are the *cochain spaces* and the elements of C^n are the degree n *cochains*. The d^n are the *differentials*. We tend to drop the index on the differentials – and so just write $d^2 = 0$ for instance.

Maps: a *(homo)morphism*, or chain map, $f : C \to D$ of complexes is a family of homomorphisms $\{f^n : C^n \to D^n\}_{n\in\mathbb{Z}}$ that commute with the differentials on C and D, i.e. all the squares in

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"] & C^n \arrow[r, "d^n"] \arrow[d, "f^n"] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & D^{n-1} \arrow[r, "e^{n-1}"] & D^n \arrow[r, "e^n"] & D^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

commute. The morphism is injective/surjective/bijective (and hence an isomorphism in this last case) when all of the f^n are.

Cohomology: for each n we have $\text{im}(d^{n-1}) \subseteq \ker(d^n)$ and the family of quotients

$$HC = H^*C := \left\{ H^nC = \frac{\ker(C^n \xrightarrow{d^n} C^{n+1})}{\text{im}(C^{n-1} \xrightarrow{d^{n-1}} C^n)} \right\}_{n\in\mathbb{Z}}$$

is called the *cohomology* of the complex. The cochains in $\ker(d^n)$ are called (degree n) *cocycles* and those in $\text{im}(d^{n-1})$ are (degree n) *coboundaries*.

The fundamental problem in homological algebra is to compute the cohomology of complexes.

If $f : C \to D$ is a morphism of complexes then there is an induced map $Hf : H^*C \to H^*D$ consisting of $Hf = \{Hf^n\}_{n\in\mathbb{Z}}$ with $Hf^n : H^nC \to H^nD$ the map

$$Hf^n : x + \mathrm{im}(d^{n-1}) \mapsto f^n(x) + \mathrm{im}(e^{n-1})$$

where d is the differential on C and e the differential on D. This mapped is well defined because f commutes with the two differentials.

The *Euler characteristic* is given by

$$\chi H^*C := \sum_n (-1)^n \dim H^nC,$$

and it is an easy exercise to show that $\chi H^*C = \sum_n (-1)^n \dim C^n$.

Subcomplexes and quotients: the complex C is a *subcomplex* of D if C^n is a subspace of D^n for all n, and $d(C^n) \subset C^{n+1}$, where d is the differential on D. Equivalently the inclusion $C \hookrightarrow D$ is a map of complexes.

The *quotient* D/C has $(D/C)^n := D^n/C^n$ and differential induced by that on D, i.e. an element $x + C^n \in D^n/C^n$ is sent to $dx + C^{n+1} \in D^{n+1}/C^{n+1}$. The induced differential is well defined precisely because C is a subcomplex.

One might expect/hope that $H^n(D/C)$ is just H^nD/H^nC, but it turns out that the relationship between these three cohomologies is more complicated than that. See the paragraph on long exact sequences below.

If $f : C \to D$ is a morphism of complexes then $\ker f$ is the subcomplex of C with $(\ker f)^n := \ker(f^n : C^n \to D^n)$ and $\mathrm{im}\, f$ is the subcomplex of D with $(\mathrm{im}\, f)^n := \mathrm{im}(f^n : C^n \to D^n)$. All the homomorphism theorems then carry straight over to complexes (e.g. $C/\ker f \cong \mathrm{im}\, f \subseteq D$, $(D/B)/(C/B) \cong D/C$ for B a subcomplex of C a subcomplex of D, etc).

Sums: If C and D are complexes then their *direct sum* $C \oplus D$ looks like:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{n-1} & \xrightarrow{d^{n-1}} & C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & \cdots \\
 & & \oplus & & \oplus & & \oplus & & \\
\cdots & \longrightarrow & D^{n-1} & \xrightarrow{e^{n-1}} & D^n & \xrightarrow{e^n} & D^{n+1} & \longrightarrow & \cdots
\end{array}$$

i.e. $(C \oplus D)^n := C^n \oplus D^n$ and the differential is the sum of the differentials on C and D. Then $H^n(C \oplus D) \cong H^nC \oplus H^nD$.

The definition of $\bigoplus_{i\in I} C_i$ is analogous. If s is a degree n cocycle, i.e. an element of $(\bigoplus_{i\in I} C_i)^n = \bigoplus_{i\in I} C_i^n$, then we write $s \cdot i$ for the coordinate of s indexed by i.

Tensor products: The *tensor product* $C \otimes D$ is slightly more complicated. Form

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & & \\
\cdots \arrow[r] & C^{p-1}\otimes D^{q+1} \arrow[r] \arrow[u] & C^{p}\otimes D^{q+1} \arrow[r] \arrow[u] & C^{p+1}\otimes D^{q+1} \arrow[r] \arrow[u] & \cdots & \\
\cdots \arrow[r] & C^{p-1}\otimes D^{q} \arrow[r, "d^{p-1}\otimes 1"] \arrow[u] & C^{p}\otimes D^{q} \arrow[r, "d^{p}\otimes 1"] \arrow[u, "1\otimes e^{q}"'] & C^{p+1}\otimes D^{q} \arrow[r] \arrow[u] & \cdots & (C\otimes D)^{n+1} \\
\cdots \arrow[r] & C^{p-1}\otimes D^{q-1} \arrow[r] \arrow[u] & C^{p}\otimes D^{q-1} \arrow[r] \arrow[u, "1\otimes e^{q-1}"'] & C^{p+1}\otimes D^{q-1} \arrow[r] \arrow[u] & \cdots & \\
 & \vdots \arrow[u] & \vdots \arrow[u] & (C\otimes D)^{n-1} \quad \vdots \arrow[u] & (C\otimes D)^{n} &
\end{tikzcd}$$

called, for obvious reasons, a double complex. All the squares commute, but using a Jedi sign-trick and replacing the $1 \otimes e^q$'s by $(-1)^p(1 \otimes e^q)$, each square acquires exactly one -1 sign, and so the squares anticommute. Define the cochain space in degree n to be

$$(C \otimes D)^n = \bigoplus_{p+q=n} C^p \otimes D^q$$

the sum over the cochain spaces in the double complex lying on the line of slope -1 with equation $x+y = n$. The differential $(C\otimes D)^{n-1} \to (C\otimes D)^n$ is the sum of all the maps between the lines $x+y = n-1$ and $x+y = n$, a differential as $(d\otimes 1)^2 = 0 = (1\otimes e)^2$ and by the anti-commuting of the squares. $H^*(C\otimes D)$ turns out to be the obvious thing, albeit for slightly non-obvious reasons (see the paragraph below on derived functors).

Shifts: If D is the complex with D^k a 1-dimensional space – a copy of the ground field F – and all other cochain groups 0, then the *shifted* complex $C[k] := C \otimes D$ has degree n cochain space equal to C^{n-k}. Thus $C[k]$ is just the complex C with everything (including differentials) shifted k units to the right. The homology gets shifted too: $H^nC[k] \cong H^{n-k}C$.

Graded point of view: in these notes the cochain spaces of our complexes turn out to be *graded* spaces. From this point of view it can be convenient to think about complexes and their cohomologies as graded spaces too. Thus a complex C is a graded space $\bigoplus_{n\in\mathbb{Z}} C^n$ equipped with a degree 1 map $d : C \to C$ satisfying $d^2 = 0$; the cohomology H^*C is also a graded space $\bigoplus_{n\in\mathbb{Z}} H^nC$.

Exactness: A sequence $A \to B \to C$ is *exact* at B if the image in B of the map $A \to B$ equals the kernel in B of the map $B \to C$. Here, A, B and C can be (vector) spaces or complexes.

For example, exactness at B in the following

$$0 \to B \to 0, \quad 0 \to B \to C, \quad A \to B \to 0$$

are equivalent to $B = 0$, the map $B \to C$ being injective, and the map $A \to B$ being surjective. Exactness at A and B in

$$0 \to A \to B \to 0$$

is equivalent to the map $A \to B$ being an isomorphism. Exactness at C^n in the complex

$$\cdots \to C^{n-1} \to C^n \to C^{n+1} \to \cdots$$

is equivalent to $H^n C = 0$, and so on.

Short exact sequences: are sequences of the form

$$0 \to A \to B \to C \to 0 \tag{1}$$

that are exact at A, B and C. Identifying A with its image in B (which we can do as $A \to B$ is injective), we get that $B \to C$ induces an isomorphism $B/A \cong C$.

Long exact sequences: are one of the most useful basic constructions in homological algebra. A short exact sequence of *complexes* (1) induces a *long exact sequence* of cohomology spaces:

$$\cdots \to H^{n-1}C \to H^n A \to H^n B \to H^n C \to H^{n+1} A \to \cdots$$

(i.e. the sequence is exact at every point). The maps $H^n A \to H^n B$ and $H^n B \to H^n C$ are those induced by $A \to B$ and $B \to C$. The maps $H^{n-1}C \to H^n A$ are called *connecting homomorphisms*, and their definition can be found in [Rot09, Wei94].

When A is a subcomplex of B we have

$$\cdots \to H^{n-1}(B/A) \to H^n A \to H^n B \to H^n(B/A) \to H^{n+1} A \to \cdots$$

If $0 \to A \to B \to C \to 0$ is a short exact sequence of complexes, and the cohomology of A vanishes everywhere, i.e. $H^n A = 0$ for all n, then the long exact sequence implies that the maps

$$H^n B \to H^n C$$

are isomorphisms for all n, i.e. that $H^* B \cong H^* C$. Similarly, if $H^n C = 0$ for all n then $H^* A \cong H^* B$.

Derived functors: suppose $\mathcal{F}$ is a functor that sends chain complexes to chain complexes. A natural question to then ask is: how does the homology of the complex $\mathcal{F}(C)$ relate to the homology of the complex C?

The answer, when there is one, is very roughly the following: there is a new functor $\mathcal{F}^1$, called the first derived functor of $\mathcal{F}$, and for each n a short exact sequence relating $H^n\mathcal{F}(C)$ and $\mathcal{F}(H^n C)$ in such a way that if $\mathcal{F}^1$ vanishes then $H^n\mathcal{F}(C) \cong \mathcal{F}(H^n C)$. So the values of the derived functor $\mathcal{F}^1$ provide obstructions to the answer of the question above being a straight-forward one. Replacing $\mathcal{F}$ by $\mathcal{F}^1$ gives a second derived functor $\mathcal{F}^2$, and so on, for an infinite family of derived functors of $\mathcal{F}$. (It turns out that Khovanov homology can be interpreted in terms of derived functors.)

The case we need is where $\mathcal{F}$ is the functor $(-) \otimes D$ for some fixed complex D and the short exact sequence relating the various ingredients is called the Künneth formula. As we are dealing with vector spaces the upshot is that

$$H^*(C \otimes D) \cong H^* C \otimes H^* D \Rightarrow H^n(C \otimes D) \cong \bigoplus_{p+q=n} H^p C \otimes H^q D$$

The topological side: a pair of morphisms $f, g : C \to D$ are *homotopic*, written $f \simeq g$, when there is a *homotopy* $h = \{h^n\}_{n\in\mathbb{Z}}$ with $h^n : C^{n+1} \to D^n$, such that $f - g = hd + eh$, i.e.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^n \arrow[r, "d"] \arrow[d, "f-g"] \arrow[dl, "h"'] & C^{n+1} \arrow[r] \arrow[d] \arrow[dl, "h"] & \cdots \\
\cdots \arrow[r] & D^{n-1} \arrow[r, "e"] & D^n \arrow[r] & D^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

C and D are *homotopy equivalent* if there are morphisms $f : C \rightleftarrows D : g$ with $fg \simeq \mathrm{id}$ and $gf \simeq \mathrm{id}$.

References

Also not encyclopedic. There is Khovanov's original paper [Kho00] and the surveys [BN02, Tur14, Tur06], all of which are very readable and well worth reading. We have cherry-picked from [BN02, Tur14, Tur06] in our treatment. For background on smooth manifolds see [Lee13]; the books [CS98, Koc04, Lic97, Oht02, Rol90] are good references for knots, links, cobordisms and the category of links, and [Dol95, Rot09, Wei94] for algebraic topology and homological algebra. The first chapter of [FLM88] is useful for graded spaces.

[BN02] Dror Bar-Natan, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. **2** (2002), 337–370 (electronic).

[CS98] J. Scott Carter and Masahico Saito, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs, vol. 55, American Mathematical Society, Providence, RI, 1998. MR1487374

[Dol95] Albrecht Dold, *Lectures on algebraic topology*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition. MR1335915

[FLM88] Igor Frenkel, James Lepowsky, and Arne Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press, Inc., Boston, MA, 1988.

[HN13] Matthew Hedden and Yi Ni, *Khovanov module and the detection of unlinks*, Geom. Topol. **17** (2013), no. 5, 3027–3076.

[Kho00] Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426.

[Koc04] Joachim Kock, *Frobenius algebras and 2D topological quantum field theories*, London Mathematical Society Student Texts, vol. 59, Cambridge University Press, Cambridge, 2004. MR2037238

[KM11] P. B. Kronheimer and T. S. Mrowka, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. **113** (2011), 97–208, DOI 10.1007/s10240-010-0030-y.

[Lee13] John M. Lee, *Introduction to smooth manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR2954043

[Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR1472978

[Oht02] Tomotada Ohtsuki, *Quantum invariants*, Series on Knots and Everything, vol. 29, World Scientific Publishing Co., Inc., River Edge, NJ, 2002. A study of knots, 3-manifolds, and their sets. MR1881401

[Rol90] Dale Rolfsen, *Knots and links*, Mathematics Lecture Series, vol. 7, Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original. MR1277811

[Rot09] Joseph J. Rotman, *An introduction to homological algebra*, 2nd ed., Universitext, Springer, New York, 2009.

[Tur14] Paul Turner, *A Hitchhikers Guide to Khovanov Homology* (2014), available at `arXiv:1409.6442`.

[Tur06] ———, *Five Lectures on Khovanov Homology* (2006), available at `math.GT/0606464v1`.

[Wat07] Liam Watson, *Knots with identical Khovanov homology*, Algebr. Geom. Topol. **7** (2007), 1389–1407, DOI 10.2140/agt.2007.7.1389. MR2350287

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.